BASIC CONSTRUCTIONS IN MOTIVIC HOMOTOPY THEORY

Abstract. Using explicit description of functors amongst (motivic) spaces and (motivic) spectra, we obtain some interesting equivalences which showcases the benefit of working with motivic spaces/spectra. The main such application would be the construction of $\mathbb{A}^1$-degree of Morel and Cazanave and we will compute it for some examples. As an interesting application of $\mathbb{A}^1$-degree, we will end with counting lines over a smooth cubic surface over $\operatorname{char} \mathbf{k} \neq 2$ via quadratic forms, as done in [KW21].

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1. Motivic spaces and spectra - revisited

We wish to give few descriptions of the functors involved in constructing the four categories $\mathcal{S}\mathrm{pc}, \mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{pc}_{\mathbf{k}}$ and $\mathcal{S}\mathrm{p}_{\mathbf{k}}$. First recall the construction of ∞-category of spectra.

$$\mathcal{S}\mathrm{pc} \underset{U}{\overset{(-)_+}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathrm{pc}_* \underset{\Omega}{\overset{\Sigma \simeq S^1 \wedge -}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathrm{pc}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathrm{p}.$$

For the motivic side, we have (for $\mathbf{k}$ arbitrary field):

$$\mathcal{S}\mathrm{m}_{\mathbf{k}} \xhookrightarrow{\text{Yon.}} \mathcal{P}(\mathcal{S}\mathrm{m}_{\mathbf{k}}) \underset{\text{incl.}}{\overset{L_{\mathbb{A}^1}}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathrm{pc}_{\mathbf{k}} \underset{U}{\overset{(-)_+}{\underset{\perp}{\rightleftarrows}}} (\mathcal{S}\mathrm{pc}_{\mathbf{k}})_* \underset{\Omega^\infty_{\mathbb{P}^1}}{\overset{\Sigma^\infty_{\mathbb{P}^1}}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathrm{p}_{\mathbf{k}}.$$

Recall the following:

(1) $\mathcal{S}\mathrm{pc}_{\mathbf{k}}$ is the category of functors $F : \mathcal{S}\mathrm{m}_{\mathbf{k}}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{pc}_*$ which satisfies $\mathbb{A}^1$-invariance and maps Nisnevich squares to Cartesian squares.

(2) $\mathcal{P}(\mathcal{S}\mathrm{m}_{\mathbf{k}})$ has smash product given by

$$(F \wedge G)(U) := F(U) \wedge G(U).$$

(3) Motivic localization $L_{\mathbb{A}^1}$ is compatible with smash product, that is, the smash product of two motivic space F and G is given by

$$F \wedge G := L_{\mathbb{A}^1}(U \mapsto F(U) \wedge G(U)).$$

(4) There are adjunctions across spaces and motivic spaces and spectra and motivic spectra:

$$\mathcal{S}\mathrm{pc} \underset{\mathrm{ev}_{\mathrm{Spec}(\mathbf{k})}}{\overset{c}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathrm{pc}_{\mathbf{k}} \qquad \mathcal{S}\mathrm{p} \underset{R}{\overset{c_{\mathrm{st}}}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathrm{p}_{\mathbf{k}}$$

Date: November 23, 2026.

where $c(X) := L_{\mathbb{A}^1}(\text{const. presheaf at } X) \simeq (U \mapsto \mathrm{Map}(\pi_0(U), X))$ and $\mathrm{ev}_{\mathrm{Spec}(\mathbf{k})}$ is the evaluation of a motivic space at $\mathrm{Spec}\,(\mathbf{k})$. This is the way in which we are interpreting ordinary spaces to be motivic spaces as well, and we will, for a space X, use the same notation X in motivic categories as well. On the other hand, c_{st} and R are difficult to define, but they satisfy the conditions

$$c_{\mathrm{st}}(\mathbb{S}^n) = \mathbb{S}^n$$

and $R(E)$ is the functor obtained from universal property via the $\otimes$-functor $X \mapsto \Sigma^\infty_+ X(\mathbf{k})$. This, by adjoint property satsfies

$$\pi_n R(E) = [\mathbb{S}^n, R(E)] \simeq [c_{\mathrm{st}}(\mathbb{S}^n), E] \simeq [\mathbb{S}^n, E] = \pi_{n,0}(E).$$

(5) (Suspension and loop): The tensor-hom adjunction holds for any object $A \in (\mathcal{S}\mathrm{pc}_{\mathbf{k}})_*$ since it is presentable ∞-category and $\wedge$ preserves small colimits (adjoint functor theorem, HTT, Corollary 5.5.2.9). In particular, we have $\Omega_A(F) := \underline{\mathrm{Hom}}\,(A, F)$ in $\mathcal{P}(\mathcal{S}\mathrm{m}_{\mathbf{k}})$. In-fact, we have

$$\Omega_A F(U) = \underline{\mathrm{Hom}}\,(A, F)(U) \simeq \mathrm{Map}_*(U_+, \underline{\mathrm{Hom}}\,(A, F)) \simeq \mathrm{Map}_*(A \wedge U_+, F).$$

by Yoneda lemma. In-fact, by using the compatibility of colimits and Map_*, we get that

$$\Omega_A F(U) \simeq \mathrm{fib}\,(F(A \times U) \to F(\text{pt.} \times U))\,.$$

Something similar is true of $A \wedge -$ as well. Indeed,

$$A \wedge F \simeq \mathrm{cofib}\,(F \to *)\,.$$

Note that cofiber and fiber exists in $(\mathcal{S}\mathrm{pc}_{\mathbf{k}})_*$ because it is pointed.

(6) On $(\mathcal{S}\mathrm{pc}_{\mathbf{k}})_*$, there are three candidates of "suspension-loop" that we can stabilize:

$$(\mathcal{S}\mathrm{pc}_{\mathbf{k}})_* \underset{\Omega_{S^1}}{\overset{S^1\wedge-}{\rightleftarrows}} (\mathcal{S}\mathrm{pc}_{\mathbf{k}})_* \qquad (\mathcal{S}\mathrm{pc}_{\mathbf{k}})_* \underset{\Omega_{\mathbb{G}_m}}{\overset{\mathbb{G}_m\wedge-}{\rightleftarrows}} (\mathcal{S}\mathrm{pc}_{\mathbf{k}})_* \qquad (\mathcal{S}\mathrm{pc}_{\mathbf{k}})_* \underset{\Omega_{\mathbb{P}^1}}{\overset{\mathbb{P}^1\wedge-}{\rightleftarrows}} (\mathcal{S}\mathrm{pc}_{\mathbf{k}})_*.$$

We chose to stabilize $\mathbb{P}^1$, but by Example 2.1, it turns out that it is same as stabilizing S^1 and $\mathbb{G}_m$ as well!

(7) (Universal property of $\mathcal{S}\mathrm{p}_{\mathbf{k}}$). Composing with the functor $X \mapsto \Sigma^\infty_{\mathbb{P}^1, L_{\mathbb{A}^1}} h_X$ gives a fully-faithful embedding

$$\mathcal{F}\mathrm{un}^{L,\otimes}(\mathcal{S}\mathrm{p}_{\mathbf{k}}, \mathcal{C}) \longrightarrow \mathcal{F}\mathrm{un}^{\otimes}(\mathcal{S}\mathrm{m}_{\mathbf{k}}, \mathcal{C})$$

whose essential image consists of functors $F : \mathcal{S}\mathrm{m}_{\mathbf{k}} \to \mathcal{C}$ which are $\mathbb{A}^1$-invariant, maps Nisnevich to co-Cartesian squares and $F(\emptyset) = 0$.

2. Working with motivic spaces (and spectra)

We see some examples of computations in the category of motivic spaces $\mathcal{S}\mathrm{pc}_{\mathbf{k}}$ and $\mathcal{S}\mathrm{p}_{\mathbf{k}}$.

Example 2.1 ($\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ in $\mathcal{S}\mathrm{pc}_{\mathbf{k}}$)**.** Observe that $\mathbb{P}^1$ has the following Nisnevich square in $\mathcal{P}_*(\mathcal{S}\mathrm{m}_{\mathbf{k}})$

$$\begin{array}{ccc} (\mathbb{G}_m, 1) & \xrightarrow{t} & (\mathbb{A}^1, 1) \\ {\scriptstyle 1/t}\downarrow & \lrcorner & \downarrow \\ (\mathbb{A}^1, 1) & \longrightarrow & (\mathbb{P}^1, \infty) \end{array}.$$

But in the category of motivic spaces, this is co-Cartesian. Since $\mathbb{A}^1 \simeq *$, we get

$$\begin{array}{ccc} (\mathbb{G}_m, 1) & \xrightarrow{t} & * \\ {\scriptstyle 1/t}\downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & (\mathbb{P}^1, \infty) \end{array},$$

that is, $\mathbb{P}^1 \simeq \Sigma\,\mathbb{G}_m = S^1 \wedge \mathbb{G}_m$.

Example 2.2 ($\mathbb{P}^1 \simeq \mathbb{A}^1/\mathbb{A}^1 - 0$ in $\mathcal{S}\mathrm{pc}_{\mathbf{k}}$)**.** Continuing the diagram in the previous example, we have in $\mathcal{S}\mathrm{pc}_{\mathbf{k}}$:

$$\begin{array}{ccc} (\mathbb{A}^1-0,1) & \xrightarrow{t} & (\mathbb{A}^1,1) \\ {\scriptstyle 1/t}\downarrow & \lrcorner & \downarrow \\ (\mathbb{A}^1,1) & \longrightarrow & (\mathbb{P}^1,\infty)\cdot \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathbb{P}^1/\mathbb{A}^1 \end{array}$$

As composition of pushouts is again a pushout, therefore, we get

$$\mathbb{A}^1/\mathbb{A}^1 - 0 \simeq \mathbb{P}^1/\mathbb{A}^1 \simeq \mathbb{P}^1.$$

Example 2.3 ($\mathbb{A}^n - 0 \simeq S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$ in $\mathcal{S}\mathrm{pc}_{\mathbf{k}}$)**.** We prove this by induction. For $n = 1$, this is immediate. For general n, observe that $\mathbb{A}^{n-1} - 0$ can be covered by open sets $U_i = \{x_i \neq 0\}$, $1 \leq i \leq n$. Let $V_1 = U_2 \cup \cdots \cup U_n$. Then we have a Nisnevich square in $\mathcal{P}(\mathcal{S}\mathrm{m}_{\mathbf{k}})$:

$$\begin{array}{ccc} U_1 \cap V_1 & \longrightarrow & U_1 \\ \downarrow & \lrcorner & \downarrow \\ V_1 & \longrightarrow & \mathbb{A}^n - 0 \end{array}.$$

Note But when mapped in $\mathcal{S}\mathrm{pc}_{\mathbf{k}}$, we see that

$$\begin{array}{ccc} (\mathbb{A}^1-0)\times(\mathbb{A}^{n-1}-0) \cong U_1 \cap V_1 & \longrightarrow & U_1 \cong (\mathbb{A}^1-0)\times\mathbb{A}^{n-1} \\ \downarrow & \rangle & \downarrow \\ \mathbb{A}^1 \times (\mathbb{A}^{n-1}-0) \cong V_1 & \longrightarrow & \mathbb{A}^n - 0 \end{array}.$$

But once we map this to $\mathcal{S}\mathrm{pc}_{\mathbf{k}}$, we instead get the co-Cartesian square

$$\begin{array}{ccc} (\mathbb{A}^1-0)\times(\mathbb{A}^{n-1}-0) & \longrightarrow & (\mathbb{A}^1-0)\times\mathbb{A}^{n-1} \simeq \mathbb{A}^1 - 0 \\ \downarrow & & \downarrow \\ \mathbb{A}^{n-1} - 0 \simeq \mathbb{A}^1 \times (\mathbb{A}^{n-1}-0) & \longrightarrow & \mathbb{A}^n - 0 \end{array}$$

Thus, $\mathbb{A}^n - 0 \simeq (\mathbb{A}^{n-1} - 0) * (\mathbb{A}^1 - 0)$. Since we have for any spaces X, Y that $X * Y \simeq \Sigma(X \wedge Y)$, therefore we get

$$\mathbb{A}^n - 0 \simeq \Sigma(\mathbb{A}^1 - 0) \wedge (\mathbb{A}^{n-1} - 0) \simeq \Sigma\,\mathbb{G}_m \wedge (S^{n-2} \wedge \mathbb{G}_m^{n-1}) \simeq S^{n-1} \wedge \mathbb{G}_m^n,$$

as we needed. Since S^n in $\mathcal{S}\mathrm{p}$ (or $\mathcal{S}\mathrm{p}_{\mathbf{k}}$) becomes $\mathbb{S}^n$ (or $\mathbb{S}^{n,0}$, in the notation of next example), therefore we get in $\mathcal{S}\mathrm{p}_{\mathbf{k}}$ the following:

$$\mathbb{A}^n - 0 \simeq \mathbb{S}^{2n-1,n}.$$

Example 2.4 (Motivic sphere spectrum)**.** Let $\mathbb{S}^n := \Sigma^n \mathbb{S}$. Then define

$$\mathbb{S}^{a,b} := \mathbb{S}^{a-2b} \wedge (\mathbb{P}^1)^{\wedge b} \simeq \mathbb{S}^{a-b} \wedge \mathbb{G}_m^{\wedge b}.$$

Thus, from what we have seen so far, we get

- $\mathbb{S} = \mathbb{S}^{0,0}$
- $\mathbb{G}_m \simeq \mathbb{S}^{1,1}$
- $\mathbb{P}^1 \simeq \mathbb{S}^{2,1}$
- $\mathbb{A}^n - 0 \simeq \mathbb{S}^{2n-1,n}$

3. Realizations, quadratic forms & $\mathbb{A}^1$-degree

Note that we have realization functors

$$\mathfrak{R}_{\mathbf{k}} : \mathcal{S}\mathrm{p}_{\mathbf{k}} \longrightarrow \mathcal{S}\mathrm{p}$$

obtained from the universal property via the functor $\mathcal{S}\mathrm{m}_{\mathbf{k}} \longrightarrow \mathcal{S}\mathrm{p}$ given by $X \mapsto \Sigma^\infty_+ X(\mathbf{k})$. Indeed, this functor is $\mathbb{A}^1$-invariant and maps Nisnevich squares to co-Cartesian squares. In-particular, we have the real and complex realizations $\mathfrak{R}_{\mathbb{R}}$ and $\mathfrak{R}_{\mathbb{C}}$.

Example 3.1 ($\mathfrak{R}_{\mathbb{C}}(\mathbb{S}^{a,b}) \simeq \mathbb{S}^a$)**.** Indeed,

$$\mathfrak{R}_{\mathbb{C}}(\mathbb{S}^{a-b} \wedge \mathbb{G}_m^b) = \mathfrak{R}_{\mathbb{C}}(\mathbb{S}^{a-b}) \wedge \mathfrak{R}_{\mathbb{C}}(\mathbb{G}_m^b) \simeq \mathbb{S}^{a-b} \wedge \left(\Sigma^\infty_+ \mathbb{G}_m(\mathbb{C})\right)^{\wedge b} \simeq \mathbb{S}^{a-b} \wedge (\mathbb{S}^1)^{\wedge b} \simeq \mathbb{S}^a.$$

Example 3.2 ($\mathfrak{R}_{\mathbb{R}}(\mathbb{S}^{a,b}) \simeq \mathbb{S}^{a-b}$)**.** Indeed,

$$\mathfrak{R}_{\mathbb{R}}(\mathbb{S}^{a-b} \wedge \mathbb{G}_m^b) \simeq \mathbb{S}^{a-b} \wedge (\Sigma^\infty_+ \mathbb{G}_m(\mathbb{R}))^{\wedge b} \simeq \mathbb{S}^{a-b} \wedge (\Sigma^\infty_+ S^0)^{\wedge b} \simeq \mathbb{S}^{a-b} \wedge (\S)^{\wedge b} \simeq \mathbb{S}^{a-b}.$$

Example 3.3 ($\mathbb{A}^n/\mathbb{A}^n - 0 \simeq \mathbb{S}^{2n,n}$)**.** By definition

$$\mathbb{A}^n/\mathbb{A}^{n-1} - 0 = \mathrm{cofib}(\mathbb{A}^n - 0 \to \mathbb{A}^n) \simeq \mathrm{cofib}(\mathbb{A}^n - 0 \to *) \simeq \Sigma(\mathbb{A}^n - 0) \simeq \Sigma\mathbb{S}^{2n-1,n} \simeq \mathbb{S}^{2n,n}.$$

Definition 3.4. Let $\varphi : S^n \to S^n$ be a smooth map of spheres. The *Brouwer degree of* φ is defined to be the integer k such that $\varphi^* : H^n)(S^n;\mathbb{Z}) \cong \mathbb{Z} \to H^n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ is given by multiplication by k. This is also obtained by a *local degree formula*

$$\deg \varphi = \sum_{x \in \varphi^{-1}(y)} \operatorname{sgn} \det d\varphi_x$$

for any $y \in S^n$ which is a regular value of φ.

Construction 3.5 (Brouwer degree)**.** Let $f : (\mathbb{P}^1, \infty) \longrightarrow (\mathbb{P}^1, \infty)$ be a pointed map. We get the maps

$$f(\mathbb{R}) : \mathbb{P}^1(\mathbb{R}) = S^1 \longrightarrow \mathbb{P}^1(\mathbb{R}) = S^1$$
$$f(\mathbb{C}) : \mathbb{P}^1(\mathbb{C}) = S^2 \longrightarrow \mathbb{P}^2(\mathbb{C}) = S^2.$$

Hence we have self-maps of spheres. As

$$\mathfrak{R}_{\mathbb{C}}\, \Sigma^\infty_+ f : \Sigma^\infty_+ \mathbb{P}^1(\mathbb{C}) \simeq \mathbb{S} \longrightarrow \Sigma^\infty_+ \mathbb{P}^1(\mathbb{C}) \simeq \mathbb{S},$$

and same for $\mathfrak{R}_{\mathbb{R}}\, \Sigma^\infty_+ f$, we get that both

$$\mathfrak{R}_{\mathbb{R}}\, \Sigma^\infty_+ f, \mathfrak{R}_{\mathbb{C}}\, \Sigma^\infty_+ f \in [\mathbb{S}, \mathbb{S}] = \pi_0(\mathbb{S}) = \varinjlim_j \pi_j(S^j) \cong \mathbb{Z}.$$

So we may think of these two maps as certain integers, and indeed these integers are Brouwer degrees of $f(\mathbb{R})$ and $f(\mathbb{C})$, respectively.

It turns out the Brouwer degrees $\mathfrak{R}_{\mathbb{R}}\, \Sigma^\infty_+ f$ and $\mathfrak{R}_{\mathbb{C}}\, \Sigma^\infty_+ f$ both can be obtained from a single quadratic form.

Construction 3.6 (Bézout form)**.** Fix any $\mathbf{k}$ with $\operatorname{char} \mathbf{k} \neq 2$. For $f = g/h : (\mathbb{P}^1, \infty) \longrightarrow (\mathbb{P}^1, \infty)$ a pointed map, i.e. $n = \deg g > \deg h$, define

$$\frac{g(X)h(Y) - g(Y)h(X)}{X - Y} =: \sum_{1 \leq p,q \leq n} c_{p,q} X^{p-1} Y^{q-1}$$

and $C = (c_{p,q})$ is a symmetric invertible matrix, i.e. it corresponds to a quadratic form $B(f)$ (assuming $\operatorname{char} \mathbf{k} \neq 2$), this is the Bézout form of f.

Remark 3.7. Given a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ by $[s : t] \mapsto [G(s,t) : H(s,t)]$ where G, H are homogeneous of same degree, we get $F = G/H$. De-homogenizing ($s = 1$ or $t = 1$) and simplifying, we get $f = g/h$ as the rational function. Conversely, beginning from $f = g/h$, we may homogenize numerator and denominator to the same degree n, giving us $G(s,t)$ and $H(s,t)$, which becomes the coordinate map.

Example 3.8. Consider $f : \mathbb{P}^1 \to \mathbb{P}^1$ given by $[s : t] \mapsto [s^2 : t^2]$. The function in this case is $f(s) = s^2$. The Bézout form of this is

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

thus the quadratic form q_f is rank 2 given by $B(f)(x, y) = 2xy$, which diagonalizes to $2x^2 - 2y^2$, therefore giving us

$$B(f) = \langle 2, -2 \rangle \in \mathrm{GW}(\mathbf{k}).$$

Note that

- Rank of $B(f) = 2 = \mathfrak{R}_{\mathbb{C}} \Sigma^\infty_+ f$,
- Signature of $B(f) = 0 = \mathfrak{R}_{\mathbb{R}} \Sigma^\infty_+ f$.

Theorem 3.9 (Morel-Cazanave)**.** *For any field* $\mathbf{k}$*, the homomorphism*

$$\begin{aligned} \mathrm{End}_*(\mathbb{P}^1) &\longrightarrow \mathrm{GW}(\mathbf{k}) \\ f &\longmapsto B(f) \end{aligned}$$

factors through an isomorphism

$$\deg^{\mathbb{A}^1} : \pi_{0,0}(\mathbb{S}) \longrightarrow \mathrm{GW}(\mathbf{k}).$$

It is this $\mathbb{A}^1$-degree that is the key in the following discussion.

4. Lines on a smooth cubic surface

Let $\mathbf{k}$ be an arbitrary field of characteristic $\neq 2$. Consider a cubic surface $V \subset \mathbb{P}^3_{\mathbf{k}}$. A line $l \subset V$ is a closed point of $\mathbb{G}(1, 3)$ such that $l \subset V_L := V \otimes_{\mathbf{k}} L$ where $L = \kappa(l)$.

For a line $l \subset V_L$, we observe that the subspace $T \subset H^0(\mathbb{P}^3_L; \mathcal{O}_{V_L}(1))$ of linear polynomials on $\mathbb{P}^3_L$ vanishing on l gives a linear system $(T, \mathcal{I}_l(1))$ which is base-point free (Lemma 10, [**?**]). Thus, we get a 2:1 mapping $\pi : V_L \to \mathbb{P}^1$ which is a Galois morphism and then restricting the unique non-trivial monodromy of this map to $l \subset V_L$, we get an involution $i_l : l \to l$, which is the Segre involution. The type of a line l is then defined to be

$$\mathrm{Type}(l) := \langle -1 \rangle \cdot \deg^{\mathbb{A}^1}(i_l).$$

Finally, for a finite field extension $L/\mathbf{k}$, the trace map is defined by

$$\begin{aligned} \mathrm{Tr}_{L/\mathbf{k}} : \mathrm{GW}(L) &\longrightarrow \mathrm{GW}(\mathbf{k}) \\ \beta &\longmapsto \mathrm{tr}_{L/\mathbf{k}} \circ \beta \end{aligned}$$

where $\mathrm{tr}_{L/\mathbf{k}} : L \to \mathbf{k}$ is the usual trace map of a finite extension. A line l is hyperbolic if $\mathrm{Type}(l)$ is $\langle 1 \rangle$, otherwise it is said to be elliptic.

Theorem 4.1 (Kass-Wickelgren)**.** *Let* $\mathbf{k}$ *be a field with* $\mathrm{char}\,\mathbf{k} \neq 2$*. Let* V *be a smooth cubic surface in* $\mathbb{P}^3_{\mathbf{k}}$*. Then we have*

$$\sum_{h \in L^\times/(L^\times)^2} (\#\text{ of lines on } V \text{ of type } h) \cdot \mathrm{Tr}_{L/\mathbf{k}}\langle h \rangle = 15\langle 1 \rangle + 12\langle -1 \rangle.$$

In other words, in $\mathrm{GW}(\mathbf{k})$ *we have the equality*

$$\sum_{l \subset V} \mathrm{Tr}_{\kappa(l)/\mathbf{k}}(\mathrm{Type}(l)) = 15\langle 1 \rangle + 12\langle -1 \rangle.$$

Example 4.2. Take rank of above equality to get over any algebraically closed field of characteristic 0 we have number of lines = 27. Take signature over $\mathbb{R}$, to get that no. of elliptic - no. of hyperbolic lines = 3. Take discriminant over $\mathbb{F}_q$ with q odd to obtain no. of elliptic lines on $V = 0 \mod 2$.